SYNCHRONIZATION IN A SYSTEM OF ESSENTIALLY NONLINEAR OBJECTS WITH A SINGLE DEGREE OF FREEDOM

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This work is concerned with the analysis of the appearance of a single frequency oscillation in a system of dynamic objects with a single degree of freedom of a determined type under the action of weak intercoupling. Different approaches to the solution of the synchronization problem are considered, and the regions of their applicability are indicated. The necessary and sufficient conditions for the stability of synchronous oscillation are given for a system of essentially nonlinear different objects. For the particular case of almost identical objects, these conditions coincide with the generalized integral stability criterion [1 and 2]. The general statement of the problem of synchronization of dynamic systems, numerous examples of synchronization which can be encountered in nature or in technology, and also an exhaustive bibliography, can be found in the work of Blekhman [1].

1. Let us consider a system composed of n dynamic objects having a single degree of freedom, and positions determined by the generalized coordinates $q_1, \ldots, q_n$. We shall assume that the manner in which the generalized coordinates are introduced, is dependent of the nature of the coupling between the objects. Thus the generalized coordinate q_i must be considered as the generalized partial coordinate of the ith object, without regard for the presence or absence of a coupling.

Furthermore we shall assume that by examination of the coupled system, we can introduce the coupling parameter μ which characterizes the degree of distorsion brought by the coupling to the motion of the object. Without making any special assumptions on the magnitude of the coupling parameter, we shall suppose that it is sufficiently small.

The coupling between the objects does not introduce further degrees of freedom, and, in the general case, gives to the objects a periodical action of frequency ν, external to the system. The partial objects, i.e. the objects in the absence of interaction, are self-contained and represent systems of material points subjected to stationary coupling.

With the given assumptions, the generalized Lagrange function of the coupled system has the form

$$L = \sum_{i=1}^{n} L_i (q_i, q_i^{\cdot}) + \mu L_0 (q_1, q_1^{\cdot}, \ldots, q_n, q_n^{\cdot}, \nu t) + \mu^2 \ldots \qquad (1.1)$$

In Expression (1.1), by virtue of the generality of the introduction of the generalized coordinates, the partial Lagrangian L_i is independent of the coupling parameter, and the Lagrangian L_0 is only a function of the generalized partial coordinates and velocities of the system, and of the nondimensional time $\tau = \nu t$.

Finally, we shall assume that the coupling between the objects has a purely conservative character with the accuracy up to the order μ^2, and furthermore, that all nonpotential forces in the system do not depend explicitly upon the time.

Thus, a nonnegligible fraction of the generalized nonpotential force does not depend upon the coupling factor and has a partial character

$$Q_i = Q_i (q_i, q_i^{\cdot}).$$

The generalized force Q_i characterizes the inflow and outflow of external energy which gives to the object an autooscillating character. In the absence of coupling, only this force stabilizes the energy level, at which the motion of the object occurs. The generalized pulses

$$p_i = \frac{\partial L}{\partial q_i^{\cdot}} = \frac{\partial L_i}{\partial q_i^{\cdot}} + \mu \frac{\partial L_0}{\partial q_i^{\cdot}} + \mu^2 \ldots \qquad (i = 1, \ldots, n) \qquad (1.2)$$

introduced by the expression of the total kinetic energy of the system are dependent generally upon the type of coupling. This dependence disappears (exactly up to the order μ) only in the case of potential or force coupling when $\partial L_0 / \partial q_i^{\cdot} \equiv 0$.

Therefore the generalized velocities obtained after transformation of the system (1.2) by means of the new canonical variables q_i, p_i ($i = 1, \ldots, n$) can be represented in the general case in the form of a series of the small coupling parameter

$$q_i^{\cdot} = v_i (q_i, p_i) + \mu v_i^{(1)} (q_1, p_1, \ldots, q_n, p_n, \tau) + \mu^2 \ldots$$
$$p_i = \frac{\partial L_i (q_i, v_i)}{\partial v_i} \qquad (i = 1, \ldots, n) \qquad (1.3)$$

We shall substitute the series (1.3) in the generalized Hamilton function of a coupled system

$$H = \sum_{i=1}^{n} p_i q_i^{\cdot} - L = \sum_{i=1}^{n} H_i (q_i, p_i) - \mu L_0 (q_1, v_1, \ldots, q_n, v_n, \tau) + \mu^2 \ldots \qquad (1.4)$$

where the partial Hamiltonian of the ith object is

$$H_i = p_i v_i (q_i, p_i) - L_i (q_i, v_i) \qquad (i = 1, \ldots, n) \qquad (1.5)$$

Thus, the equation of motion of a coupled system of objects in the canonical form is

$$q_i^{\cdot} - \frac{\partial H_i}{\partial p_i} = -\mu \frac{\partial L_0}{\partial p_i} + \mu^2 \ldots, \quad p_i^{\cdot} + \frac{\partial H_i}{\partial q_i} - Q_i = \mu \frac{\partial L_0}{\partial q_i} + \mu^2 \ldots$$
$$(i = 1, \ldots, n) \tag{1.6}$$

2. If no potential forces Q_i are applied, the motion of isolated objects ($\mu = 0$) is described by a system of equations which can be broken down into n inependent purely conservative subsystems

$$q_i^{\circ\cdot} = \frac{\partial H_i (q_i^\circ, p_i^\circ)}{\partial p_i^\circ}, \qquad p_i^{\circ\cdot} = -\frac{\partial H_i (q_i^\circ, p_i^\circ)}{\partial q_i^\circ} \qquad (i = 1, \ldots, n) \tag{2.1}$$

Each subsystem (2.1) in some region G_i of the partial phase plane (q, p) has the general solution

$$q_i^\circ = x_i (\psi_i, s_i), \qquad p_i^\circ = y_i (\psi_i, s_i) \tag{2.2}$$

of a libration or rotation type with a 2π period for its fundamental rapidly rotating phase

$$\psi_i = \omega_i (s_i) t + \alpha_i \tag{2.3}$$

in the sense that

$$x_i (\psi_i + 2\pi, s_i) = x_i (\psi_i, s_i) + \gamma_i, \qquad y_i (\psi_i + 2\pi, s_i) = y_i (\psi_i, s_i) \tag{2.4}$$

Furthermore, the γ_i do not depend upon the values s_i, and the Hamiltonian function of the partial object $H_i(q_i, p_i)$, as it can be easily shown, is γ_i-periodic in q_i for $\gamma_i \neq 0$.

The general solution of (2.2) inside the region G_i depends in a continuous manner upon the arbitrary phase shift α_i, and upon the parameter of energy, the integral of force s_i introduced by the relation

$$s_i = \frac{1}{2\pi} \int_0^{2\pi} y_i (\psi_i, s_i) \frac{\partial x_i (\psi_i, s_i)}{\partial \psi_i} d\psi_i \tag{2.5}$$

The integral of force which is single-valued and continuous inside G_i is related to the integral of energy $H_i(x_i, y_i) = h_i(s_i)$.

The frequency of motion (angular velocity) of the object, which is introduced by the relation

$$\omega_i (s_i) = \frac{dh_i (s_i)}{ds_i} \tag{2.6}$$

in the general case depends upon the energetic level and can change inside G_i within a certain finite (or infinite) interval (frequency range)

$$\omega_i^{(1)} < \omega_i < \omega_i^{(2)}.$$

As we consider the system (1.6) which describes the motion of the coupled objects, we shall assume that the nonpotential functions $Q_i(q_i, v_i)$, the Lagrangian of coupling $L_0 (q_1, v_1, \ldots, q_n, v_n, \tau)$, as well as all the components of order equal or superior to μ^2, are bounded, analytic with respect to all their arguments, γ_i-periodic in their variables q_i and 2π-periodic in τ inside a region of the $2n$-dimensional phase space of the system, such that the pairs (q_i, p_i) are located inside G_i.

3. Let us make in (1.6) the canonical transformation of variables

$$q_i = x_i(\varphi_i, J_i) \qquad p_i = y_i(\varphi_i, J_i) \qquad (i = 1, \ldots, n) \tag{3.1}$$

which is possible, since by virtue of (2.5)

$$\frac{\partial x_i}{\partial \varphi_i}\frac{\partial y_i}{\partial J_i} - \frac{\partial y_i}{\partial \varphi_i}\frac{\partial x_i}{\partial J_i} = 1 \tag{3.2}$$

We come to the following specific system with respect to the canonical variables "force-angle"

$$J_i^{\cdot} - \frac{\partial x_i}{\partial \varphi_i} Q_i = \mu \frac{\partial L_0}{\partial \varphi_i} + \mu^2 \ldots, \qquad \varphi_i^{\cdot} - \omega_i(J_i) + \frac{\partial x_i}{\partial J_i} Q_i = -\mu \frac{\partial L_0}{\partial J_i} + \mu^2 \ldots$$
$$(i = 1, \ldots, n) \tag{3.3}$$

In the consideration of system (3.3), it is necessary to keep in mind, as a consequence of the small coupling, the synchronous mode is possible for the system, if, in the system consisting of the partial objects isolated from one another it is possible to have motions which are from both the quantitative and the qualitative points of view, close to the real synchronous ones on a finite but sufficiently large interval of time. Then the generating system can always be chosen such that it really admits a solution of the required type (*).

Let the frequency ranges of the objects be small

$$\omega_i^{(2)} - \omega_i^{(1)} = 0(\mu) \qquad \text{or} \qquad \omega_i(J_i) = \lambda_i + \mu\,\omega_i'(J_i) \tag{3.4}$$

then, the system obtained from (3.3) for $\mu = 0$ has, in the general case, a multiple frequency mode characterized by the partial frequencies $\lambda_1, \ldots, \lambda_n$. Its energetic level is stabilized by the action of the nonpotential forces $Q_1, \ldots, Q_n$. In such a case the synchronous generating solution can occur only when the condition

$$\lambda_1 = \ldots = \lambda_n = \nu \tag{3.5}$$

is fulfilled.

If the frequency ranges of the objects are not small, the nonpotential forces in the partial objects begin to behave as frequency stabilizers. In order to have a synchronous solution of the generating system, some rigorous conditions must be imposed on them.

In a system of essentially different objects, the synchronization is possible if the stabilizing action of the nonpotential forces does not have an effect on the generating approximation, i.e. if it is small and, consequently, does not exceed the synchronizing actions transmitted by the coupling. Thus, it was natural to assume that the inflow and outflow of energy into the partial object

$$Q_i(q_i, q_i^{\cdot}) = \mu F_i(q_i, q_i^{\cdot}) \tag{3.6}$$

are small.

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*) Considering here the generating system, we suppose that the complementary terms of the order μ can be separated from the left-hand side of Equations (3.3).

Therefore, the generating system of equations (3.3) coincides with (2.1) and admits a synchronous solution if the intersection of the frequency ranges of the separate generating objects

$$(\omega^{(1)}, \omega^{(2)}) = \bigcap_{i=1}^{n} (\omega_i^{(1)}, \omega_i^{(2)}) \tag{3.7}$$

is not empty and includes the frequency ν of the external force.

This last case is very important indeed, because by applying to it the methods of the perturbation theory, we could trace the drift of the motion frequencies of the different objects while the synchronous mode builds up. On the contrary, the consideration of the isochronous generating approximation, for instance for the degenerated quasilinear formulation, leads essentially to the problem of the building up of phase shifts, amplitudes, and so on.

The different particular cases of coupling of autooscillating systems by means of small internal forces, are related to the study of the isochronous, generally linear generating approximation already considered in the literature [3 and 4]. Thus, obviously, we could always register the absence of a synchronous mode for the system in the case of essentially different partial frequencies.

We shall consider the problem of the interaction of essentially nonlinear objects when the system can adjust itself to the external force frequency in a sufficiently large range determined by the equality (3.7). On the basis of this study, we shall write the equations of the motion of a coupled system of objects with respect to the new "phase-frequency" variables, which, according to (3.6) has the form

$$\omega_i^{\cdot} = \frac{\mu}{k_i(\omega_i)}\left(\frac{\partial x_i}{\partial \varphi_i} F_i + \frac{\partial L_0}{\partial \varphi_i}\right) + \mu^2 \ldots$$

$$\varphi_i^{\cdot} - \omega_i = -\frac{\mu}{k_i(\omega_i)}\left(\frac{\partial x_i}{\partial \omega_i} F_i + \frac{\partial L_0}{\partial \omega_i}\right)\mu^2 \ldots \qquad (i = 1, \ldots, n) \tag{3.8}$$

The transformation to "phase-frequency" variables, more characteristic of synchronization problems, does not contain singularities since everywhere in the phase domain of the system

$$k_i(\omega_i) = \frac{dJ_i(\omega_i)}{d\omega_i} \doteq \frac{1}{\omega_i}\frac{dh_i(\omega_i)}{d\omega_i} = O(1) \tag{3.9}$$

4. First of all, generalizing somewhat the problem, we shall consider the interaction of essentially nonlinear, almost conservative objects, described by the following system with a multidimensional rapidly rotating phase

$$\omega_i^{\cdot} = \mu Y_i(\varphi, \omega, \tau) + \mu^2 \ldots, \qquad \varphi_i^{\cdot} = \omega_i + \mu X_i(\varphi, \omega, \tau) + \mu^2 \ldots \tag{4.1}$$

Here

$$Y_i(\varphi, \omega, \tau) = Y_i(\varphi_1, \ldots, \varphi_n; \omega_1, \ldots, \omega_n; \tau)$$

$$X_i(\varphi, \omega, \tau) = X_i(\varphi_1, \ldots, \varphi_n; \omega_1, \ldots, \omega_n; \tau) \qquad (i = 1, \ldots, n) \tag{4.2}$$

are analytic in a certain domain G of the phase space of the system, have a 2π period with respect to the rapidly rotating phase φ_i $(i = 1, \ldots, n)$ and to the nondimensional time $\tau = \nu t$.

The analysis of the system (4.1) is essentially possible on the basis of the generalized averaging method developed lately in the work of Volosov [5]. However some mathematical difficulties arise. Therefore we shall confine ourselves to the analysis of the synchronous mode in the system and its small neighborhood for a sufficiently small μ by using Poincaré's method.

The generating system

$$\omega_i^{\circ\cdot} = 0, \quad \varphi_i^{\circ\cdot} = \omega_i^\circ \quad (i = 1, \ldots, n) \tag{4.3}$$

has a general solution

$$\omega_i^\circ = \nu_i, \quad \varphi_i^\circ = \nu_i t + \alpha_i \tag{4.4}$$

which depends on $2n$ arbitrarily chosen variables ν_i and α_i.

The generating solution is characterized by

$$\nu_1 = \ldots = \nu_n = \nu \tag{4.5}$$

and the phase shifts are single roots of the system [6]

$$P_i\,(\alpha_1, \ldots, \alpha_n) = \frac{1}{2\pi}\int_0^{2\pi} Y_i\,[\tau + \alpha_1, \ldots, \tau + \alpha_n;\, \nu_1, \ldots, \nu_n;\, \tau]\,d\tau = 0 \quad (i = 1, \ldots, n) \tag{4.6}$$

The fulfilment of conditions (4.5) and (4.6) guarantees the existence of a solution of the system (4.1) having a $\frac{2\pi}{\nu}$ period in t. Successive periodic approximations to this solution can be sought in the form of the series

$$\omega_i = \nu + \mu\omega_i^{(1)} + \mu^2 \ldots, \qquad \varphi_i = \tau + \alpha_i + \mu\varphi_i^{(1)} + \mu^2 \ldots \tag{4.7}$$

Analyzing the local stability of the synchronous motion (4.7) we shall write the equations of the variations for the system (4.1) considered

$$\frac{dU_i}{dt} = \mu\sum_{j=1}^{n}\left[\left(\frac{\partial Y_i}{\partial\varphi_j}\right)V_j + \left(\frac{\partial Y_i}{\partial\omega_j}\right)U_j\right] + \mu^2 \ldots$$

$$\frac{dV_i}{dt} = U_i + \mu\sum_{j=1}^{n}\left[\left(\frac{\partial X_i}{\partial\omega_j}\right)V_j + \left(\frac{\partial X_i}{\partial\omega_j}\right)U_j\right] + \mu^2 \ldots \tag{4.8}$$

Here the parentheses mean that the corresponding quantity is calculated for the generating approximation. Let us introduce the new variables

$$U_i = e^{a(\mu)t}u_i, \qquad V_i = e^{a(\mu)t}v_i \tag{4.9}$$

where $a(\mu)$ is the characteristic exponent which we write in the present case [7] as a series

$$a\,(\mu) = a_1\mu^{1/2} + a_2\mu + a_3\mu^{3/2} + \mu^2 \ldots \tag{4.10}$$

Now, the purpose of the problem is the determination of the existence conditions of the periodic solution of the system

$$\frac{du_i}{dt} = -\mu^{1/2}a_1u_i + \mu\left\{-a_2u_i + \sum_{j=1}^{n}\left[\left(\frac{\partial Y_i}{\partial\varphi_j}\right)v_j + \left(\frac{\partial Y_i}{\partial\omega_j}\right)u_j\right]\right\} - \mu^{3/2}a_3u_i + \mu^2 \ldots \tag{4.11}$$

$$\frac{dv_i}{dt} = u_i - \mu^{1/2} a_1 v_i + \mu \Big\{ - a_2 v_i + \sum_{j=1}^{n} \Big[\Big(\frac{\partial X_i}{\partial \varphi_j}\Big) v_j + + \Big(\frac{\partial X_i}{\partial \omega t}\Big) u_j \Big] \Big\} - \mu^{3/2} a_3 v_i + \mu^2 \ldots \qquad (4.11)\ \text{cont.}$$

We shall seek for successive periodic approximations to the solutions of the system (4.11) in the form of series

$$\begin{aligned} u_i &= u_i^\circ + \mu^{1/2} u_i^{(1)} + \mu u_i^{(2)} + \mu^{3/2} \ldots \\ v_i &= v_i^\circ + \mu^{1/2} v_i^{(1)} + \mu v_i^{(2)} + \mu^{3/2} \ldots \end{aligned} \qquad (4.12)$$

The general periodic solution of the zero approximation equation

$$du_i^\circ / dt = 0, \qquad dv_i^\circ / dt = u_i^\circ \qquad (4.13)$$

depends upon n arbitrary constants M_i and has the form

$$u_i^\circ = 0, \qquad v_i^\circ = M_i \qquad (4.14)$$

the first approximation equation

$$du_i^{(1)} / dt = 0, \qquad dv_i^{(1)} / dt = u_i^{(1)} - a_1 M_i \qquad (4.15)$$

has always the periodical solution

$$u_i^{(1)} = a_1 M_i, \qquad v_i^{(1)} = N_i \qquad (4.16)$$

which depends on $2n$ constants M_i and N_i.

The periodic solution of the second approximation equations (4.17)

$$\frac{du_i^{(2)}}{dt} = \sum_{j=1}^{n} \Big[\Big(\frac{\partial Y_i}{\partial \varphi_j}\Big) - a_1^2 \delta_{ij}\Big] M_j \qquad \frac{dv_i^{(2)}}{dt} = u_i^{(2)} - a_1 N_i + \sum_{j=1}^{n} \Big[\Big(\frac{\partial X_i}{\partial \varphi_j}\Big) - a_2 \delta_{ij}\Big] M_j$$

exists if

$$\sum_{j=1}^{n} M_j \Big(\frac{\partial P_i}{\partial \alpha_j} - a_1^2 \delta_{ij}\Big) = 0 \qquad (i = 1, \ldots, n) \qquad (4.18)$$

and has the form

$$\begin{aligned} u_i^{(2)} &= a_1 N_i + a_2 M_i + D_i - a_1^2 M_i t + \sum_{j=1}^{n} M_j \frac{\partial}{\partial \alpha_j} \int_0^t (Y_i)\, dt \\ v_i^{(2)} &= C_i + D_i t - a_1^2 M_i \frac{t^2}{2} \sum_{j=1}^{n} M_j \frac{\partial}{\partial \alpha_j} \int_0^t \Big[(X_i) + \int_0^t (Y_i)\, dt \Big] dt \end{aligned} \qquad (4.19)$$

In Equation (4.19) the constants C_i are arbitrary and

$$D_i = a_1^2 \frac{\pi}{\nu} M_i - \sum_{j=1}^{n} M_j \frac{\nu}{2\pi} \frac{\partial}{\partial \alpha_j} \int_0^{2\pi/\nu} \Big[(X_i) + \int_0^t (Y_i)\, dt \Big] dt \qquad (4.20)$$

The condition of nontriviality of the solution of the system (4.18)

$$\Big| \frac{\partial P_i}{\partial \alpha_j} - a_1^2 \delta_{ij} \Big| = 0 \qquad (4.21)$$

is used for determination of the first approximations to the real characteristic exponents.

The existence condition of the periodic solution of the first group of the third approximation equations

$$\frac{du_i^{(3)}}{dt} = \sum_{j=1}^{n}\left[\left(\frac{\partial Y_i}{\partial \varphi_j}\right) - a_1^2\delta_{ij}\right]N_j - 2a_1a_2M_i -$$
$$-a_1^2M_i\left(\frac{\pi}{\nu} - t\right) + a_1\sum_{j=1}^{n} M_j\frac{\nu}{2\pi}\frac{\partial}{\partial\alpha_j}\int_0^{2\pi/\nu}\left[(X_i) + \int_0^t (Y_i)\,dt\right]dt - \qquad (4.22)$$
$$-a_1\sum_{j=1}^{n} M_j\frac{\partial}{\partial\alpha_j}\int_0^t (Y_j)\,dt + a_1\sum_{j=0}^{n} M_j\left(\frac{\partial Y_i}{\partial\omega_j}\right)$$

can be brought to the form

$$\sum_{j=1}^{n}\left(\frac{\partial P_i}{\partial\alpha_j} - a_1^2\delta_{ij}\right)N_j = a_1\left[2a_2M_i - \sum_{j=1}^{n}\left(\frac{\partial R_i}{\partial\alpha_j} + \frac{\partial P_i}{\partial\nu_j}\right)M_j\right] \qquad (4.23)$$

Here

$$R_i(\alpha_1,\ldots,\alpha_n) = \frac{1}{2\pi}\int_0^{2\pi} X_i\,[\tau+\alpha_1,\ldots\tau+\alpha_n;\ \nu_1,\ldots,\nu_n;\ \tau]\,d\tau \qquad (4.24)$$

The presence of the periodic solution of the second group of the third approximation equations is guaranteed by the choice of proper integration constants when the system (4.22) is integrated.

The second approximation to the characteristic exponents are determined from the conditions of solvability of the heterogeneous system (4.23) with respect to the unknown N_j

$$a_2 = \left(2\sum_{i=1}^{n} M_iM_i^*\right)^{-1}\sum_{i,j=1}^{n}\left(\frac{\partial R_i}{\partial\alpha_j} + \frac{\partial P_i}{\partial\nu_j}\right)M_jM_i^* \qquad (4.25)$$

The numbers M_i^* ($i = 1,\ldots,n$) are a solution of the system which is coupled with the system (4.18)

$$\sum_{j=1}^{n} M_j^*\left(\frac{\partial P_j}{\partial\alpha_i} - a_1^2\delta_{ij}\right) = 0 \qquad (i = 1,\ldots,n) \qquad (4.26)$$

The synchronous mode of the system is asymptotically stable if

$$\operatorname{Re} a_1 = 0, \quad \operatorname{Re} a_2 < 0$$

5. Applying now the results obtained in the previous section, in the study of the synchronous solutions of the system (3.8), we write the basic equations which determine the generating phase shifts in the form

$$P_i(\alpha_1,\ldots,\alpha_n) = \frac{1}{k_i(\nu)}\left(f_i + \frac{\partial\Lambda}{\partial\alpha_i}\right) = 0 \qquad (i = 1,\ldots,n) \qquad (5.1)$$

where

$$f_i = \frac{1}{2\pi}\int_0^{2\pi}(F_i)\frac{\partial x_i}{\partial\varphi_i}\,d\varphi_i, \qquad \Lambda(\alpha_1,\ldots,\alpha_n) = \frac{1}{2\pi}\int_0^{2\pi}(L_0)\,d\tau \qquad (5.2)$$

is the average work on a period of the nonpotential force of the partial object and the integral of force of the coupling in the generating approximation.

The determinant of the system for the determination of the first approximation to the characteristic exponents

$$\sum_{j=1}^{n}\left(\frac{l_{ij}}{k_i(\nu)}-a_1^2\delta_{ij}\right)M_j=0 \qquad \left(l_{ij}=\frac{\partial^2\Lambda}{\partial\alpha_i\partial\alpha_j}\right) \tag{5.3}$$

can be written in the following symmetrical form:

$$\left(\prod_{i=1}^{n}k_i(\nu)\right)^{-1}\left|l_{ij}-k_i(\nu)\,a_1^2\delta_{ij}\right|=0 \tag{5.4}$$

A sufficient, but obviously nonnecessary condition for the roots of the determinant (5.4) ($\mathrm{Im}\ a_1^2 = 0$) to be real is that at least one of the quadratic forms corresponding to the matrix

$$\|l_{ij}\|$$

or $\mathrm{diag}\,(k_1(\nu), \ldots, k_n(\nu))$, be positive or negative definite. The presence of a complex root is possible if there is a simultaneous change in the sign of these forms; it is characterized by an identical transformation into zero of the quantity

$$\sum_{i=1}^{n}k_i(\nu)\,|M_i|^2$$

computed for the given root.

In the case of identical, purely conservative objects ($k_1(\nu) = \ldots = k_n(\nu) = k$) the necessary condition of stability $a_1^2 < 0$ reduces to the requirement of an extremum (maximum or minimum depending upon the sign of k) for the integral of force of the coupling Λ and leads to the formulation of a generalized integral stability criterion. The intgral criterion can also be generalized in the case of an internal synchronization for a system of almost identical objects of the considered type [2]

The solution of the system

$$\sum_{j=1}^{n}\left(\frac{l_{ij}}{k_j(\nu)}-a_1^2\delta_{ij}\right)M_j^*=0 \tag{5.5}$$

coupled with (5.3) can obviously be obtained from the solution of the system (5.3) in the following manner:

$$M_j^* = k_j(\nu)\,M_j \tag{5.6}$$

If, furthermore, we take into consideration that

$$R_i(\alpha_1, \ldots, \alpha_n) = -\frac{1}{k_i(\nu)}\left(f_i^* + \frac{\partial\Lambda}{\partial\nu_i}\right) \tag{5.7}$$

where the quantities f_i^* are independent of the generating phase shifts, the expressions of the second order approximation of the characteristic exponents take the form

$$a_2=\left(2\sum_{i=1}^{n}k_i(\nu)\,M_i^2\right)^{-1}\sum_{i,j=1}^{n}\left[-\frac{\partial^2\Lambda}{\partial\alpha_i\partial\nu_j}+\frac{\partial^2\Lambda}{\partial\alpha_j\partial\nu_i}+\frac{\partial f_i}{\partial\nu_i}\delta_{ij}\right]M_iM_j$$

or

$$a_2 = \frac{1}{2}\left(\sum_{i=1}^{n} k_i(\nu) M_i^2\right)^{-1} \sum_{i=1}^{n} \frac{\partial f_i}{\partial \nu_i} M_i^2 \qquad (5.8)$$

It should be remembered that

$$\frac{\partial f_i}{\partial \nu_i} = k_i(\nu) \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial F_i}{\partial y_i}\right) d\varphi_i \qquad (5.9)$$

The fulfilment of the asymptotic stability condition $a_2 < 0$ is thus only possible if nonconserative forces are present in the system.

In a purely conservative system, the first approximations to the characteristic exponents do not change, but the second approximations become identically equal to zero.

We shall notice that in the particular degenerate case of a single object ($n = 1$) the conditions of existence and stability of a synchronous mode of the system are in agreement with the corresponding relations obtained by Kats [8].

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BIBLIOGRAPHY

1. Blekhman, I.I., Problemy sinkhronizatsii dinamicheskikh sistem (The problem of synchronization of dynamical systems). *PMM* Vol.28, № 2, 1964.

2. Nagaev, R.F., O sinkhronizatsii pochti odinakovykh dinamicheskikh sistem, blizkikh k sistemam Liapunova (The synchronization of nearly-similar dynamic systems close to Liapunov systems). *PMM* Vol.28, № 3, 1964.

3. Minorskii, N., O sinkhronizatsii (On synchronization). International Symposium on nonlinear oscillations, Institute of Mathematics, Acad. of Sciences USSR, Kiev, 1961.

4. Rubanik, V.P., O vzaimodeistvii dvukh nelineinykh avtokolebatel'nykh sistem pri nalichii malykh zapazdyvaiushchikh sil sviazi (On the interaction of two nonlinear self-oscillating systems when small delayed coupling forces are applied . Tr.seminara po teorii differentsial'nykh uravnenii s otkloniaiushchimsia argumentom (Proceedings of the seminar on the theory of differential equations with diverging argument). Izd.Univ.Druzhby Narodov (Publication of the University of the friendship among Nations). M., 1963.

5. Volosov, V.M., O vysshikh priblizheniiakh pri usrednenii (On the higher order approximations in the averaging). Dokl.Akad.Nauk SSSR, Vol.137, № 5, 1961.

6. Malkin, I.G., Nekotorye zadachi teorii nelineinykh kolebanii (Some Problems of the Theory of Nonlinear Oscillations). Gostekhizdat, M.,1956.

7. Kushul', M.Ia., O kvazigarmonicheskikh sistemakh, blizkikh k sistemam s postoiannymi koeffitsientami (On the quasiharmonic systems close to systems with constant coefficients). *PMM* Vol.22, № 4, 1958.

8. Kats, A.M., Vynuzhdennye kolebaniia nelineinykh sistem s odnoi stepen'iu svobody, blizkikh k konservativnym (Forced oscillations of nonlinear systems with one degree of freedom, close to concervative systems). *PMM* Vol.19, № 1, 1955.

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